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Generalized Lie derivations on skew elements of prime algebras with involution

Pao-Kuei Liau, Cheng-Kai Liu *

Department of Mathematics, National Changhua University of Education, Changhua 500, Taiwan

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ABSTRACT

We characterize generalized Lie derivations on skew elements of prime algebras $\mathcal{A}$ with involution, provided that $\mathcal{A}$ does not satisfy polynomial identities of low degree. The analogous result for matrix algebras is also described.

1. Introduction

Let $\mathcal{F}$ be a commutative ring with 1 and let $\mathcal{A}$ be an associative algebra over $\mathcal{F}$. For $x, y \in \mathcal{A}$, we set $[x, y] = xy - yx$ the Lie product of x and y and $x \circ y = xy + yx$ the Jordan product of x and y. Given any two subsets $\mathcal{U}$ and $\mathcal{V}$, we denote by $[\mathcal{U}, \mathcal{V}]$ the $\mathcal{F}$-submodule of $\mathcal{A}$ generated by $\{[x, y] \mid x \in \mathcal{U}, y \in \mathcal{V}\}$ and $\mathcal{U} \circ \mathcal{V}$ the $\mathcal{F}$-submodule of $\mathcal{A}$ generated by $\{x \circ y \mid x \in \mathcal{U}, y \in \mathcal{V}\}$. It is well known that an $\mathcal{F}$-algebra $\mathcal{A}$ is also a Lie $\mathcal{F}$-algebra with respect to the Lie product [,]. An $\mathcal{F}$-submodule $\mathcal{L}$ of $\mathcal{A}$ is called a Lie subalgebra if $[\mathcal{L}, \mathcal{L}] \subseteq \mathcal{L}$. An $\mathcal{F}$-submodule $\mathcal{L}$ of $\mathcal{A}$ satisfying a stronger condition $[\mathcal{L}, \mathcal{A}] \subseteq \mathcal{L}$ is called a Lie ideal of $\mathcal{A}$. Now assume that $\mathcal{A}$ is an $\mathcal{F}$-algebra with a linear involution $*$. Let $\mathcal{K} = \{x \in \mathcal{A} \mid x^* = -x\}$ be the set of skew elements in $\mathcal{A}$. Clearly, $\mathcal{K}$ is a Lie subalgebra of $\mathcal{A}$. A Lie ideal of $\mathcal{K}$ is an $\mathcal{F}$-submodule $\mathcal{L}$ of $\mathcal{K}$ satisfying $[\mathcal{L}, \mathcal{K}] \subseteq \mathcal{L}$. For instance, $[\mathcal{K}, \mathcal{K}]$ is a Lie ideal of $\mathcal{K}$. A

* Corresponding author.
E-mail addresses: d96211001@mail.ncue.edu.tw (P.-K. Liau), ckliu@cc.ncue.edu.tw (C.K. Liu).

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linear map $d : \mathcal{A} \to \mathcal{A}$ is called a derivation of $\mathcal{A}$ if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{A}$. A linear map $d : \mathcal{L} \to \mathcal{A}$ of a Lie subalgebra $\mathcal{L}$ of $\mathcal{A}$ is called a Lie derivation if $d([x, y]) = [d(x), y] + [x, d(y)]$ for all $x, y \in \mathcal{L}$. The problem of describing Lie derivations of some important Lie subalgebras of an associative algebra was posed by Herstein in 1961 [21]. The natural conjecture was that, modulo maps into the center, the Lie derivations are the derivations of associative algebras. The first result in this direction was obtained in an unpublished paper of Kaplansky in the case of matrix algebras over a field. With the presence of nontrivial idempotents, this problem had been examined by Martindale [37] for primitive algebra and by Miers [39] for von Neumann algebras. Until in 1993, Brešar [14] used the technique of functional identities to solve Herstein's problem for prime algebras in full generality without assuming the existence of nontrivial idempotents. Later, Swain [41] extended this result to skew elements of prime algebras with involution. Recently, Beidar and Chebotar [4] applied the theory of functional identities to describe the Lie derivations of Lie ideals of prime algebras. The analogous result for Lie ideals of skew elements was obtained by Beidar, Brešar, Chebotar and Martindale in [6]. Over the past few years, these results had been generalized on various rings and operator algebras (see for instance [1,2,8,15,16,33,34,36,42,44,45]). Motivated by the notion of generalized derivations, Hvala [24] initiated the study of generalized Lie derivations.

A linear map $g : \mathcal{A} \to \mathcal{A}$ is called a generalized derivation of $\mathcal{A}$ if there exists a derivation d of $\mathcal{A}$ such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in \mathcal{A}$. Basic examples are derivations, generalized inner derivations (i.e., maps of the form $x \mapsto ax + xb$ for some $a, b \in \mathcal{A}$) and the left centralizers (i.e., linear maps satisfying $g(xy) = g(x)y$ for all $x, y \in \mathcal{A}$). The notation of generalized derivations was introduced by Brešar [13] and these maps had been extensively investigated not only in algebra but also in analysis (see for instance [3,9,11,17,18,19,20,23,25,26,28,29,31,32,35,38,43,46]). A linear map $f : \mathcal{L} \to \mathcal{A}$ of a Lie subalgebra $\mathcal{L}$ of $\mathcal{A}$ is called a generalized Lie derivation if there exists a linear map $d : \mathcal{L} \to \mathcal{A}$ such that

$$f([x, y]) = f(x)y - f(y)x + xd(y) - yd(x)$$

for all $x, y \in \mathcal{L}$. In [24], Hvala gave the definition of a generalized Lie derivation and described its structure on prime algebras. Recently, this result had been extended to Lie ideals of prime algebras in [27], to Banach algebras in [30] and to nest algebras in [40]. The goal of this paper is to describe the generalized Lie derivations on Lie ideals of skew elements of prime algebras with involution. Our main result is as follows:

Theorem 1.1. *Let $\mathcal{A}$ be a prime $\mathcal{F}$-algebra of characteristic not 2, with a linear involution, with the set of skew elements $\mathcal{K}$ and with the extended centroid $\mathcal{C}$, where $\mathcal{F}$ is a subring of $\mathcal{C}$ with $1, \frac{1}{2} \in \mathcal{F}$. Let $\mathcal{L}$ be a noncentral Lie ideal of $\mathcal{K}$ and let $\mathcal{B}$ be the subalgebra of $\mathcal{A}$ generated by $\mathcal{L}$. Assume that $\mathcal{A}$ does not satisfy the standard polynomial identity of degree 36. If $f : \mathcal{L} \to \mathcal{A}$ is a generalized Lie derivation, then there exist a generalized derivation $g : \mathcal{B} \to \mathcal{A}\mathcal{C} + \mathcal{C}$ and a linear map $\zeta : \mathcal{L} \to \mathcal{C}$ such that $f(x) = g(x) + \zeta(x)$ for all $x \in \mathcal{L}$ and $\zeta([\mathcal{L}, \mathcal{L}]) = 0$.*

In the case of matrix algebras, we have

Theorem 1.2. *Let $\mathcal{A} = M_n(\mathcal{F})$, $n \geqslant 18$, be the matrix algebra over a field $\mathcal{F}$ of characteristic not 2, with a linear involution. Let $\mathcal{K}$ be the set of skew elements in $\mathcal{A}$ and let $f : \mathcal{K} \to \mathcal{A}$ be a generalized Lie derivation. Then there exist a generalized derivation $g : \mathcal{A} \to \mathcal{A}$ such that $f(x) = g(x)$ for all $x \in \mathcal{K}$.*

2. Preliminaries

The approach in this paper will be based on the useful theory of functional identities as developed by Beidar, Brešar, Chebotar and Martindale. A functional identity can be informally described as an identical relation involving elements in a ring together with set-theoretic maps. For a full account of the theory we refer the reader to the recent book [12]. Let $\mathcal{A}$ be a ring and let $\mathcal{V}$ be a subset of $\mathcal{A}$. The concept of d-free sets, where d is a positive integer, always plays an important role in the theory of functional identities (see [12]). Roughly speaking, we say that $\mathcal{V}$ is a d-free subset of $\mathcal{A}$ if every appropriate functional identity on $\mathcal{V}$ in less than d variables has only standard solutions. We omit the precise definition and instead we state a special result that is needed here.

Lemma 2.1. [12, Lemma 3.2]. *Let $d \geqslant 1$ be an integer, let $\mathcal{A}$ be a ring and let $\mathcal{V}$ be a subset of $\mathcal{A}$. Let $B_i : \mathcal{V}^{d-1} \to \mathcal{A}$, $i = 1, \ldots, d$, be maps such that*

$$\sum_{i=1}^{d} r_i B_i(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_d) = 0 \quad \text{for all} \quad r_1, r_2, \ldots, r_d \in \mathcal{V}.$$

Suppose that $\mathcal{V}$ is d-free. Then $B_i = 0$ for all $i = 1, \ldots, d$.

We make a remark that B_1 is just a constant in $\mathcal{A}$ whenever $d = 1$. We need the following fact.

Lemma 2.2. [12, Theorem 3.4]. *Let $d \geqslant 1$ be an integer, let $\mathcal{A}$ be a ring and let $\mathcal{V}, \mathcal{U}$ be subsets of $\mathcal{Q}$ with $\mathcal{V} \subseteq \mathcal{U}$. Suppose that $\mathcal{V}$ is d-free. Then $\mathcal{U}$ is also d-free.*

The following proposition is crucial.

Proposition 2.3. *Let $\mathcal{F}$ be a commutative ring with $1, \frac{1}{2}$, let $\mathcal{A}$ be an $\mathcal{F}$-algebra and let $\mathcal{R}$ be a Lie subalgebra of $\mathcal{A}$ such that $\mathcal{R} \cap (\mathcal{R} \circ \mathcal{R}) = 0$ and $xyz + zyx \in \mathcal{R}$ for all $x, y, z \in \mathcal{R}$. Let $\mathcal{L}$ be a Lie ideal of $\mathcal{R}$ and let $\mathcal{B}$ be the subalgebra of $\mathcal{A}$ generated by $\mathcal{L}$. If $f : \mathcal{L} \to \mathcal{A}$ is a linear map satisfying $f([x, y]) = f(x)y - f(y)x$ for all $x, y \in \mathcal{L}$, then there is a linear map $F : \mathcal{B} \to \mathcal{A}$ such that $F(xy) = F(x)y$ for all $x, y \in \mathcal{B}$ and $F(x) = f(x)$ for all $x \in \mathcal{L}$ provided that $\mathcal{L}$ is 8-free.*

Proof Given $x, y, z, t \in \mathcal{L}$, we have $xyz + zyx \in \mathcal{R}$ and $[xyz + zyx, t] \in \mathcal{L}$. Define the map $B : \mathcal{L}^4 \to \mathcal{A}$ by the rule

$$B(x, y, z, t) = f([xyz + zyx, t]) - f(x)yzt - f(z)yxt + f(t)xyz + f(t)zyx$$

for all $x, y, z, t \in \mathcal{L}$. Clearly, B is $\mathcal{F}$-multilinear. Note that we have the identity

$$[xyz + zyx, t] + [yzt + tzy, x] + [ztx + xtz, y] + [txy + yxt, z] = 0 \tag{1}$$

for all $x, y, z, t \in \mathcal{L}$. Using (1), we see that

$$B(x, y, z, t) + B(y, z, t, x) + B(z, t, x, y) + B(t, x, y, z) = 0 \tag{2}$$

for all $x, y, z, t \in \mathcal{L}$. Also for $x, y, z, t, s \in \mathcal{L}$,

$$\begin{aligned}
&B(x, y, z, [t, s]) \\
&\quad = f([xyz + zyx, [t, s]]) - f(x)yz[t, s] - f(z)yx[t, s] + f([t, s])xyz + f([t, s])zyx \\
&\quad = f([[xyz + zyx, t], s]) + f([t, [xyz + zyx, s]]) - f(x)yz[t, s] - f(z)yx[t, s] \\
&\qquad + (f(t)s - f(s)t)xyz + (f(t)s - f(s)t)zyx \\
&\quad = f([xyz + zyx, t])s - f(s)[xyz + zyx, t] + f(t)[xyz + zyx, s] - f([xyz + zyx, s])t \\
&\qquad - f(x)yz[t, s] - f(z)yx[t, s] + (f(t)s - f(s)t)xyz + (f(t)s - f(s)t)zyx \\
&\quad = B(x, y, z, t)s - B(x, y, z, s)t.
\end{aligned} \tag{3}$$

From (2) it follows that

$$\begin{aligned}
&B([x_1, x_2], [y_1, y_2], [z_1, z_2], [t_1, t_2]) + B([y_1, y_2], [z_1, z_2], [t_1, t_2], [x_1, x_2]) \\
&\quad + B([z_1, z_2], [t_1, t_2], [x_1, x_2], [y_1, y_2]) + B([t_1, t_2], [x_1, x_2], [y_1, y_2], [z_1, z_2]) = 0
\end{aligned}$$

for all $x_1, x_2, y_1, y_2, z_1, z_2, t_1, t_2 \in \mathcal{L}$. By (3), we have

$$\begin{aligned} & B([x_1, x_2], [y_1, y_2], [z_1, z_2], t_1)t_2 - B([x_1, x_2], [y_1, y_2], [z_1, z_2], t_2)t_1 \\ & \quad + B([y_1, y_2], [z_1, z_2], [t_1, t_2], x_1)x_2 - B([y_1, y_2], [z_1, z_2], [t_1, t_2], x_2)x_1 \\ & \quad + B([z_1, z_2], [t_1, t_2], [x_1, x_2], y_1)y_2 - B([z_1, z_2], [t_1, t_2], [x_1, x_2], y_2)y_1 \\ & \quad + B([t_1, t_2], [x_1, x_2], [y_1, y_2], z_1)z_2 - B([t_1, t_2], [x_1, x_2], [y_1, y_2], z_2)z_1 = 0 \end{aligned}$$

for all $x_1, x_2, y_1, y_2, z_1, z_2, t_1, t_2 \in \mathcal{L}$. By Lemma 2.1, we obtain

$$B([x_1, x_2], [y_1, y_2], [z_1, z_2], t_1) = 0 \tag{4}$$

for all $x_1, x_2, y_1, y_2, z_1, z_2, t_1 \in \mathcal{L}$. Next from (2), it follows that

$$\begin{aligned} & B([x_1, x_2], [y_1, y_2], [z_1, z_2], t_1) + B([y_1, y_2], [z_1, z_2], t_1, [x_1, x_2]) \\ & \quad + B([z_1, z_2], t_1, [x_1, x_2], [y_1, y_2]) + B(t_1, [x_1, x_2], [y_1, y_2], [z_1, z_2]) = 0 \end{aligned}$$

for all $x_1, x_2, y_1, y_2, z_1, z_2, t_1 \in \mathcal{L}$. Using (4) and (3), we obtain

$$\begin{aligned} & B([y_1, y_2], [z_1, z_2], t_1, x_1)x_2 - B([y_1, y_2], [z_1, z_2], t_1, x_2)x_1 \\ & \quad + B([z_1, z_2], t_1, [x_1, x_2], y_1)y_2 - B([z_1, z_2], t_1, [x_1, x_2], y_2)y_1 \\ & \quad + B(t_1, [x_1, x_2], [y_1, y_2], z_1)z_2 - B(t_1, [x_1, x_2], [y_1, y_2], z_2)z_1 = 0 \end{aligned}$$

for all $x_1, x_2, y_1, y_2, z_1, z_2, t_1 \in \mathcal{L}$. Again by Lemma 2.1, we see that

$$B([y_1, y_2], [z_1, z_2], t_1, x_1) = 0, \tag{5}$$

and

$$B(t_1, [x_1, x_2], [y_1, y_2], z_1) = 0 \tag{6}$$

for all $x_1, x_2, y_1, y_2, z_1, z_2, t_1 \in \mathcal{L}$. From (2) it follows that

$$\begin{aligned} & B([y_1, y_2], [z_1, z_2], t_1, x_1) + B([z_1, z_2], t_1, x_1, [y_1, y_2]) \\ & \quad + B(t_1, x_1, [y_1, y_2], [z_1, z_2]) + B(x_1, [y_1, y_2], [z_1, z_2], t_1) = 0 \end{aligned} \tag{7}$$

for all $x_1, x_2, y_1, y_2, z_1, z_2, t_1 \in \mathcal{L}$. Applying (5) and (6) to (7), we obtain

$$B([z_1, z_2], t_1, x_1, [y_1, y_2]) + B(t_1, x_1, [y_1, y_2], [z_1, z_2]) = 0$$

for all $x_1, y_1, y_2, z_1, z_2, t_1 \in \mathcal{L}$. By (3), we get

$$\begin{aligned} & B([z_1, z_2], t_1, x_1, y_1)y_2 - B([z_1, z_2], t_1, x_1, y_2)y_1 \\ & \quad + B(t_1, x_1, [y_1, y_2], z_1)z_2 - B(t_1, x_1, [y_1, y_2], z_2)z_1 = 0 \end{aligned}$$

for all $x_1, y_1, y_2, z_1, z_2, t_1 \in \mathcal{L}$. Using Lemma 2.1, we see that

$$B([z_1, z_2], t_1, x_1, y_1) = 0 = B(t_1, x_1, [y_1, y_2], z_1) \tag{8}$$

for all $x_1, y_1, y_2, z_1, z_2, t_1 \in \mathcal{L}$. Now from (2) it follows that

$$\begin{aligned} & B([z_1, z_2], t_1, [x_1, x_2], y_1) + B(t_1, [x_1, x_2], y_1, [z_1, z_2]) \\ & \quad + B([x_1, x_2], y_1, [z_1, z_2], t_1) + B(y_1, [z_1, z_2], t_1, [x_1, x_2]) = 0 \end{aligned}$$

for all $x_1, x_2, y_1, z_1, z_2, t_1 \in \mathcal{L}$. Applying (8) to above equality, we have

$$B(t_1, [x_1, x_2], y_1, [z_1, z_2]) + B(y_1, [z_1, z_2], t_1, [x_1, x_2]) = 0$$

for all $x_1, x_2, y_1, z_1, z_2, t_1 \in \mathcal{L}$. Hence by (3)

$$\begin{aligned} &B(t_1, [x_1, x_2], y_1, z_1)z_2 - B(t_1, [x_1, x_2], y_1, z_2)z_1 \\ &\quad + B(y_1, [z_1, z_2], t_1, x_1)x_2 - B(y_1, [z_1, z_2], t_1, x_2)x_1 = 0 \end{aligned}$$

for all $x_1, x_2, y_1, z_1, z_2, t_1 \in \mathcal{L}$. Applying Lemma 2.1 to above equality, we obtain

$$B(t_1, [x_1, x_2], y_1, z_1) = 0 \tag{9}$$

for all $x_1, x_2, y_1, z_1, t_1 \in \mathcal{L}$. By (2),

$$\begin{aligned} &B([z_1, z_2], t_1, x_1, y_1) + B(t_1, x_1, y_1, [z_1, z_2]) \\ &\quad + B(x_1, y_1, [z_1, z_2], t_1) + B(y_1, [z_1, z_2], t_1, x_1) = 0 \end{aligned}$$

for all $x_1, x_2, y_1, z_1, t_1 \in \mathcal{L}$. By (8) and (9), $B(t_1, x_1, y_1, [z_1, z_2]) = 0$ follows. Hence $B(t_1, x_1, y_1, z_1)z_2 - B(t_1, x_1, y_1, z_2)z_1 = 0$ for all $x_1, y_1, z_1, z_2, t_1 \in \mathcal{L}$. By Lemma 2.1, $B(t_1, x_1, y_1, z_1) = 0$ for all $x_1, y_1, z_1, t_1 \in \mathcal{L}$. Consequently,

$$f([xyz + zyx, t]) = f(x)yzt + f(z)yxt - f(t)xyz - f(t)zyx \tag{10}$$

for all $x, y, z, t \in \mathcal{L}$. Set $\overline{\mathcal{L}}$ to be equal to the $\mathcal{F}$-submodule of $\mathcal{A}$ generated by the subset $\{x, xyz + zyx \mid x, y, z \in \mathcal{L}\}$. Given $x, y, z \in \mathcal{L}$ and $t \in \mathcal{R}$, we have

$$[xyz + zyx, t] = [x, t]yz + zy[x, t] + x[y, t]z + z[y, t]x + xy[z, t] + [z, t]yx \in \overline{\mathcal{L}}. \tag{11}$$

This implies $\overline{\mathcal{L}}$ is a Lie ideal of $\mathcal{R}$ containing $\mathcal{L}$. Now we define a map $F : \overline{\mathcal{L}} \to \mathcal{A}$ by the rule

$$F\left(\sum_{i=1}^{n} x_i y_i z_i + z_i y_i x_i + t\right) = \sum_{i=1}^{n} f(x_i)y_i z_i + f(z_i)y_i x_i + f(t)$$

for all $x_i, y_i, z_i, t \in \mathcal{L}$. First we show that F is well-defined. Assume $\sum_{i=1}^{n} x_i y_i z_i + z_i y_i x_i + t = 0$ for $x_i, y_i, z_i, t \in \mathcal{L}$ and pick any $s \in \mathcal{L}$. Then $[\sum_{i=1}^{n} x_i y_i z_i + z_i y_i x_i, s] = -[t, s]$. So by (10), we get

$$\begin{aligned} &\left(\sum_{i=1}^{n} f(x_i)y_i z_i + f(z_i)y_i x_i\right) s - f(s)\left(\sum_{i=1}^{n} x_i y_i z_i + z_i y_i x_i\right) \\ &= f\left(\left[\sum_{i=1}^{n} x_i y_i z_i + z_i y_i x_i, s\right]\right) \\ &= -f([t, s]) \\ &= -\left(f(t)s - f(s)t\right) \\ &= -f(t)s - f(s)\left(\sum_{i=1}^{n} x_i y_i z_i + z_i y_i x_i\right). \end{aligned}$$

Thus $(\sum_{i=1}^{n} f(x_i)y_i z_i + f(z_i)y_i x_i + f(t))s = 0$ for all $s \in \mathcal{L}$. Then by Lemma 2.1, $\sum_{i=1}^{n} f(x_i)y_i z_i + f(z_i)y_i x_i + f(t) = 0$. This implies that $F(\sum_{i=1}^{n} x_i y_i z_i + z_i y_i x_i + t) = 0$ and so F is well-defined. Clearly, F is $\mathcal{F}$-linear and $F(x) = f(x)$, $F(xyz + zyx) = F(x)yz + F(z)yx$ for all $x, y, z \in \mathcal{L}$. Next we claim that $F([x, y]) = F(x)y - F(y)x$ for all $x, y \in \overline{\mathcal{L}}$. By the definition of F and (10), we already have $F([x, y]) = F(x)y - F(y)x$ and $F([xyz + zyx, t]) = F(xyz + zyx)t - F(t)(xyz + zyx)$ and $F([t, xyz + zyx]) = F(t)(xyz + zyx) - F(xyz + zyx)t$ for all $x, y, z, t \in \mathcal{L}$. Let $x, y, z, x'y'z' \in \mathcal{L}$. From (11), it follows that

$$\begin{aligned}[xyz+zyx,x'y'z'+z'y'x'] = {} & [x,x'y'z'+z'y'x']yz+zy[x,x'y'z'+z'y'x']+x[y,x'y'z'\\ & +z'y'x']z+z[y,x'y'z'+z'y'x']x+xy[z,x'y'z'+z'y'x']\\ & +[z,x'y'z'+z'y'x']yx \in \overline{\mathcal{L}}. \end{aligned} \tag{12}$$

By (12) and (10), we see that

$$\begin{aligned} F([xyz+zyx,x'y'z'+z'y'x']) = {} & F([x,x'y'z'+z'y'x']yz+zy[x,x'y'z'+z'y'x'])\\ & +F(x[y,x'y'z'+z'y'x']z+z[y,x'y'z'+z'y'x']x)\\ & +F(xy[z,x'y'z'+z'y'x']+[z,x'y'z'+z'y'x']yx)\\ = {} & F([x,x'y'z'+z'y'x'])yz+F(z)y[x,x'y'z'+z'y'x']\\ & +F(x)[y,x'y'z'+z'y'x']z+F(z)[y,x'y'z'+z'y'x']x\\ & +F(x)y[z,x'y'z'+z'y'x']+F([z,x'y'z'+z'y'x'])yx\\ = {} & F(x)(x'y'z'+z'y'x')yz-F(x'y'z'+z'y'x')xyz\\ & +F(z)y[x,x'y'z'+z'y'x']+F(x)[y,x'y'z'+z'y'x']z\\ & +F(z)[y,x'y'z'+z'y'x']x+F(x)y[z,x'y'z'+z'y'x']\\ & +F(z)(x'y'z'+z'y'x')yx-F(x'y'z'+z'y'x')zyx\\ = {} & F(xyz+zyx)(x'y'z'+z'y'x')-F(x'y'z'+z'y'x')(xyz+zyx), \end{aligned}$$

proving the claim.

Let Ω be the set of all pairs $(\mathcal{U}, f_{\mathcal{U}})$ such that

(1) $\mathcal{U}$ is a Lie ideal of $\mathcal{R}$ containing $\mathcal{L}$;
(2) $f_{\mathcal{U}} : \mathcal{U} \to \mathcal{A}$ is an $\mathcal{F}$-module map such that $f_{\mathcal{U}}([x,y]) = f_{\mathcal{U}}(x)y - f_{\mathcal{U}}(y)x$ for all $x, y \in \mathcal{U}$ and $f_{\mathcal{U}}(x) = f(x)$ for all $x \in \mathcal{L}$.

Define a relation "$\leqslant$" on Ω by $(\mathcal{U}, f_{\mathcal{U}}) \leqslant (\mathcal{V}, f_{\mathcal{V}})$ if $\mathcal{U} \subseteq \mathcal{V}$ and $f_{\mathcal{U}}(x) = f_{\mathcal{V}}(x)$ for all $x \in \mathcal{U}$. This defines a partial ordering on Ω and it follows from Zorn's lemma that Ω has a maximal element, say, $(\mathcal{U}, f_{\mathcal{U}})$. Since $\mathcal{L} \subseteq \mathcal{U}$, by Lemma 2.2, $\mathcal{U}$ is 8-free. Let $\overline{\mathcal{U}}$ be the $\mathcal{F}$-submodule of $\mathcal{A}$ generated by the subset $\{x, xyz+zyx \mid x,y,z \in \mathcal{U}\}$. In view of the proof above, we conclude that $\overline{\mathcal{U}}$ is a Lie ideal of $\mathcal{R}$ containing $\mathcal{U}$ and there exists a $\mathcal{F}$-linear map $F_{\mathcal{U}} : \overline{\mathcal{U}} \to \mathcal{A}$ such that $F_{\mathcal{U}}(x) = f_{\mathcal{U}}(x)$, $F_{\mathcal{U}}(xyz+zyx) = f_{\mathcal{U}}(x)yz + f_{\mathcal{U}}(z)yx$ for all $x,y,z \in \mathcal{U}$ and $F_{\mathcal{U}}([x,y]) = F_{\mathcal{U}}(x)y - F_{\mathcal{U}}(y)x$ for all $x, y \in \overline{\mathcal{U}}$. By the maximality of $\mathcal{U}$, $\overline{\mathcal{U}} = \mathcal{U}$ and hence $F_{\mathcal{U}} = f_{\mathcal{U}}$. Thus $\mathcal{U}$ is a Lie subalgebra of $\mathcal{A}$ and $xyz+zyx \in \mathcal{U}$ for all $x,y,z \in \mathcal{U}$. Let $\mathcal{A}'$ be the subalgebra of $\mathcal{A}$ generated by $\mathcal{U}$. Then by [5, Lemma 3.3] $\mathcal{A}' = \mathcal{U} + \mathcal{U} \circ \mathcal{U}$. Using $(x \circ y) \circ z = [[z,x],y] + [[z,y],x] + 2(xzy+yzx) \in \mathcal{U}$ for $x,y,z \in \mathcal{U}$, we have

$$\begin{aligned} f_{\mathcal{U}}((x\circ y)\circ z) = {} & f_{\mathcal{U}}([[z,x],y]) + f_{\mathcal{U}}([[z,y],x]) + 2f_{\mathcal{U}}(xzy+yzx)\\ = {} & f_{\mathcal{U}}([z,x])y - f_{\mathcal{U}}(y)[z,x] + f_{\mathcal{U}}([z,y])x - f_{\mathcal{U}}(x)[z,y] + 2(f_{\mathcal{U}}(x)zy + f_{\mathcal{U}}(y)zx)\\ = {} & (f_{\mathcal{U}}(z)x - f_{\mathcal{U}}(x)z)\,y - f_{\mathcal{U}}(y)[z,x] + (f_{\mathcal{U}}(z)y - f_{\mathcal{U}}(y)z)x - f_{\mathcal{U}}(x)[z,y]\\ & + 2(f_{\mathcal{U}}(x)zy + f_{\mathcal{U}}(y)zx)\\ = {} & f_{\mathcal{U}}(x)yz + f_{\mathcal{U}}(y)xz + f_{\mathcal{U}}(z)xy + f_{\mathcal{U}}(z)yx. \end{aligned} \tag{13}$$

We define the map $F' : \mathcal{A}' \to \mathcal{A}$ by

$$F'\left(\sum_{i=1}^{n} x_i \circ y_i + z\right) = \sum_{i=1}^{n} f_{\mathcal{U}}(x_i)y_i + f_{\mathcal{U}}(y_i)x_i + f_{\mathcal{U}}(z)$$

for $x_i, y_i, z \in \mathcal{U}$. We claim that F' is well-defined. Assume $\sum_{i=1}^n x_i \circ y_i + z = 0$. Then $-z = \sum_{i=1}^n x_i \circ y_i \in \mathcal{U} \cap \mathcal{U} \circ \mathcal{U} \subseteq \mathcal{R} \cap \mathcal{R} \circ \mathcal{R} = \{0\}$, implying $z = \sum_{i=1}^n x_i \circ y_i = 0$. Let $t \in \mathcal{U}$. Then $(\sum_{i=1}^n x_i \circ y_i) \circ t = 0$. Then by (13),

$$
\begin{aligned}
0 &= f_{\mathcal{U}}\left(\left(\sum_{i=1}^n x_i \circ y_i\right) \circ t\right) = f_{\mathcal{U}}\left(\sum_{i=1}^n ((x_i \circ y_i) \circ t)\right) \\
&= \sum_{i=1}^n (f_{\mathcal{U}}(x_i) y_i t + f_{\mathcal{U}}(y_i) x_i t + f_{\mathcal{U}}(t) x_i \circ y_i) \\
&= \left(\sum_{i=1}^n f_{\mathcal{U}}(x_i) y_i + f_{\mathcal{U}}(y_i) x_i\right) t.
\end{aligned}
$$

Thus $(\sum_{i=1}^n f_{\mathcal{U}}(x_i) y_i + f_{\mathcal{U}}(y_i) x_i)\mathcal{U} = 0$. By Lemma 2.1, $\sum_{i=1}^n f_{\mathcal{U}}(x_i) y_i + f_{\mathcal{U}}(y_i) x_i = 0$. Clearly, $f_{\mathcal{U}}(z) = 0$. This proves the claim. Clearly F' is $\mathcal{F}$-linear, $F'(x) = f_{\mathcal{U}}(x)$ and $F'(x^2) = F'(\frac{1}{2} x \circ x) = \frac{1}{2} F'(x \circ x) = f_{\mathcal{U}}(x) x = F'(x) x$ for all $x \in \mathcal{U}$. Using the identity $xy = \frac{1}{2}\left((x+y)^2 - x^2 - y^2 + [x, y]\right)$ for $x, y \in \mathcal{U}$, we have

$$
\begin{aligned}
F'(xy) &= \frac{1}{2} F'\left((x+y)^2 - x^2 - y^2 + [x, y]\right) \\
&= \frac{1}{2}\left(F'((x+y)^2) - F'(x^2) - F'(y^2) + F'([x, y])\right) \\
&= \frac{1}{2}\left(f_{\mathcal{U}}(x+y)(x+y) - f_{\mathcal{U}}(x) x - f_{\mathcal{U}}(y) y + f_{\mathcal{U}}(x) y - f_{\mathcal{U}}(y) x\right) \\
&= f_{\mathcal{U}}(x) y = F'(x) y.
\end{aligned}
\tag{14}
$$

Next using the identity $(x \circ y)z = \frac{1}{2}(x \circ [y, z] + y \circ [x, z] + (x \circ y) \circ z)$ for $x, y, z \in \mathcal{U}$ and (13), by the definition of F', we have

$$
\begin{aligned}
F'((x \circ y)z) &= \frac{1}{2} F'(x \circ [y, z] + y \circ [x, z] + (x \circ y) \circ z) \\
&= \frac{1}{2}\left(F'(x \circ [y, z]) + F'(y \circ [x, z]) + F'((x \circ y) \circ z)\right) \\
&= \frac{1}{2}\left(f_{\mathcal{U}}(x)[y, z] + f_{\mathcal{U}}([y, z]) x + f_{\mathcal{U}}(y)[x, z] + f_{\mathcal{U}}([x, z]) y + f_{\mathcal{U}}((x \circ y) \circ z)\right) \\
&= f_{\mathcal{U}}(x) yz + f_{\mathcal{U}}(y) xz = F'(x \circ y) z.
\end{aligned}
\tag{15}
$$

Recall that $\mathcal{A}' = \mathcal{U} + \mathcal{U} \circ \mathcal{U}$. By (14) and (15), we conclude that $F'(az) = F'(a)z$ for all $a \in \mathcal{A}'$ and $z \in \mathcal{U}$. Let $a \in \mathcal{A}'$ and $y, z \in \mathcal{U}$. Then $F'(a(yz + zy)) = F'((ay)z) + F'((az)y) = F'(ay)z + F'(az)y = F'(a)yz + F'(a)zy = F'(a)(yz + zy)$. This implies that $F'(ab) = F'(a)b$ for all $a, b \in \mathcal{A}'$. Since $\mathcal{B} \subseteq \mathcal{A}'$, the proof is now complete. $\square$

Lemma 2.4. *Let $\mathcal{A}$ be an algebra and let $\mathcal{L}$ be a Lie subalgebra of $\mathcal{A}$. Suppose that $f : \mathcal{L} \to \mathcal{A}$ is a generalized Lie derivation and $d : \mathcal{L} \to \mathcal{A}$ is a linear map such that $f([x, y]) = f(x)y - f(y)x + xd(y) - yd(x)$ for all $x, y \in \mathcal{L}$. Then d is the derivation of Lie algebras, that is, $d([x, y]) = [d(x), y] + [x, d(y)]$ for all $x, y \in \mathcal{L}$, provided that $\mathcal{L}$ is 3-free.*

Proof. By assumption, for $x, y, z \in \mathcal{R}$,

$$
\begin{aligned}
f([[x, y], z]) &= f([x, y])z - f(z)[x, y] + [x, y]d(z) - zd([x, y]) \\
&= (f(x)y - f(y)x + xd(y) - yd(x))z - f(z)[x, y] \\
&\quad + [x, y]d(z) - zd([x, y]).
\end{aligned}
\tag{16}
$$

Similarly, we have

$$f([[y,z],x]) = (f(y)z - f(z)y + yd(z) - zd(y))x - f(x)[y,z] + [y,z]d(x) - xd([y,z]) \quad (17)$$

and

$$f([[z,x],y]) = (f(z)x - f(x)z + zd(x) - xd(z))y - f(y)[z,x] + [z,x]d(y) - yd([z,x]). \quad (18)$$

Using the Jacobi identity $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$, the sum of (16), (17) and (18) yields

$$xB(y,z) + yB(z,x) + zB(x,y) = 0$$

where $B(x,y) = d([x,y]) - [d(x),y] - [x,d(y)]$. By Lemma 2.1, $d([x,y]) - [d(x),y] - [x,d(y)] = 0$ for all $x, y \in \mathcal{L}$. This proves the lemma. $\square$

3. Main results

Given a positive integer n, we denote by S_n the symmetric group of degree n. The standard polynomial St_n of degree n is defined as follows:

$$St_n = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$$

where $(-1)^\sigma$ is the sign of the permutation σ. Before proving our results, we make the following remark.

Remark 3.1 [12]. Let $\mathcal{A}$ be a prime algebra of characteristic not 2, with a linear involution and let $\mathcal{K}$ be the set of skew elements in $\mathcal{A}$. Then $\mathcal{K}$ is d-free if $\mathcal{A}$ does not satisfy the standard polynomial identity of degree $4d+2$ and any noncentral Lie ideal of $\mathcal{K}$ is d-free if $\mathcal{A}$ does not satisfy the standard polynomial identity of degree $4d+4$. In particular, if $\mathcal{A} = M_n(\mathcal{F})$ is the matrix algebra over a field $\mathcal{F}$ of characteristic not 2, then $\mathcal{K}$ is d-free if $n \geqslant 2d+2$.

Theorem 3.2. *Let $\mathcal{A}$ be a prime $\mathcal{F}$-algebra of characteristic not 2, with a linear involution, with the set of skew elements $\mathcal{K}$ and with the extended centroid $\mathcal{C}$, where $\mathcal{F}$ is a subring of $\mathcal{C}$ with $1, \frac{1}{2} \in \mathcal{F}$. Let $\mathcal{L}$ be a noncentral Lie ideal of $\mathcal{K}$ and let $\mathcal{B}$ be the subalgebra of $\mathcal{A}$ generated by $\mathcal{L}$. In case $\mathcal{L} = \mathcal{K}$, we assume that $\mathcal{A}$ does not satisfy the standard polynomial identity of degree 34. In case $\mathcal{L} \neq \mathcal{K}$, we assume that $\mathcal{A}$ does not satisfy the standard polynomial identity of degree 36. If $f : \mathcal{L} \to \mathcal{A}$ is a generalized Lie derivation, then there exist a generalized derivation $g : \mathcal{B} \to \mathcal{AC} + \mathcal{C}$ and a linear map $\zeta : \mathcal{L} \to \mathcal{C}$ such that $f(x) = g(x) + \zeta(x)$ for all $x \in \mathcal{L}$ and $\zeta([\mathcal{L}, \mathcal{L}]) = 0$.*

Proof. Clearly, $\mathcal{K} \cap \mathcal{K} \circ \mathcal{K} = 0$ and $xyz + zyx \in \mathcal{K}$ for all $x, y, z \in \mathcal{K}$. By assumption and Remark 3.1, $\mathcal{L}$ is 8-free. Let $d : \mathcal{L} \to \mathcal{A}$ be the linear map such that $f([x,y]) = f(x)y - f(y)x + xd(y) - yd(x)$ for all $x, y \in \mathcal{L}$. By Lemma 2.4 and [6, Theorem 1.8], there exist a derivation $\delta : \mathcal{B} \to \mathcal{AC} + \mathcal{C}$ and a linear map $\zeta : \mathcal{L} \to \mathcal{C}$ such that $d(x) = \delta(x) + \zeta(x)$ for all $x \in \mathcal{L}$ and $\zeta([\mathcal{L}, \mathcal{L}]) = 0$. Set $f' = f - d$. Then $f'([x,y]) = f'(x)y - f'(y)x$ for all $x, y \in \mathcal{L}$. By Proposition 2.3, there exists a linear map $F : \mathcal{B} \to \mathcal{A}$ such that $F(xy) = F(x)y$ for all $x, y \in \mathcal{B}$ and $F(x) = f'(x)$ for all $x \in \mathcal{L}$. Now it is easy to check that the map $g = F + \delta : \mathcal{B} \to \mathcal{AC} + \mathcal{C}$ is a generalized derivation and $f(x) = g(x) + \zeta(x)$ for all $x \in \mathcal{L}$. This completes the proof. $\square$

Clearly, Theorem 1.1 is an immediate consequence of Theorem 3.2 and Remark 3.1. We note that if $\mathcal{A}$ is a noncommutative simple algebra of characteristic not 2, then the subalgebra generated by the skew elements $\mathcal{K}$ coincides with $\mathcal{A}$ unless $\mathcal{A}$ is 4-dimensional over its center [22, Theorem 2.1.10]. It is well known that if $\mathcal{A} = M_n(\mathcal{F})$ is the matrix algebra over a field $\mathcal{F}$ of characteristic not 2, with linear involution $*$, then $\mathcal{K} = [\mathcal{K}, \mathcal{K}]$ provided that $n \geqslant 5$. Thus Theorem 1.2 follows directly from Theorem 3.2 and Remark 3.1. Finally, by the standard theory of polynomial identities, we have

Theorem 3.3. *Let $\mathcal{A}$ be a simple $\mathcal{F}$-algebra of characteristic not 2, with linear involution $*$, with skew elements $\mathcal{K}$ and with center $\mathcal{Z}(\mathcal{A})$, where $\mathcal{F}$ is a subring of $\mathcal{Z}(\mathcal{A})$ with $1, \frac{1}{2} \in \mathcal{F}$. Suppose that $\dim_{\mathcal{Z}(\mathcal{A})}\mathcal{A} \geqslant 36$ and $f : \mathcal{K} \to \mathcal{A}$ is a generalized Lie derivation, then there exist a generalized derivation $g : \mathcal{A} \to \mathcal{A}$ and a linear map $\zeta : \mathcal{K} \to \mathcal{Z}(\mathcal{A})$ such that $f(x) = g(x) + \zeta(x)$ for all $x \in \mathcal{K}$ and $\zeta([\mathcal{K}, \mathcal{K}]) = 0$.*

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